11/7 ==highorder derivatives.==

$U\subset\mathbb{R}^n$ קבוצה פתוחה.

$F:\mathbb{R}^n\longrightarrow\mathbb{R}^1$, נניח ש- $\frac{\partial f}{\partial x_j}$ גזירות ב-$U$, $1\le j\le n$, אז מגדירים

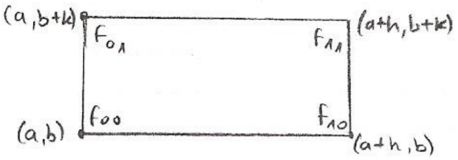
$$\frac{\partial^2 f}{\partial x_i x_j}=\frac{\partial}{x_i}\left(\frac{\partial f}{\partial x_j}\right)\qquad 1\le i,j\le n$$

$$\frac{\partial^k f}{\partial x_{i_1}\dots\partial x_{i_k}}$$

האם נגזרות חלקיות תלויות בסדר?

למשל - האם $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$ ?

$F(x,y)$:




לכן תלוי,

$h,k>0$

$$\frac{\partial F}{\partial x}(a,b)\approx\frac{F(a+h,b)-F(a,b)}{h}=\frac{F_{10}-F_{00}}{h}$$

$$\frac{\partial F}{\partial x}(a,b+k)\approx\frac{F_{11}-F_{01}}{h}$$

$$\frac{\partial^2 F}{\partial y\partial x}(a,b)\approx\frac{1}{k}\left(\frac{\partial F}{\partial x}(a,b+k)-\frac{\partial F}{\partial x}(a,b)\right)=\frac{F_{11}-F_{01}-F_{10}+f_{00}}{kh}$$

$$\frac{\partial^2 f}{\partial x\partial y}(a,b)\approx\frac{F_{11}-F_{10}-F_{01}+F_{00}}{kh}$$

$g:\mathbb{R}^2\longrightarrow\mathbb{R}^1$ חסומה. נגדיר -

$$F(x,y)=xy\,g(x,y)$$

$$(a,b)=0$$

$$\frac{\partial F}{\partial x}(0,y)=\lim_{x\to0}\frac{F(x,y)-F(0,y)}{x}=\lim_{x\to0}y\cdot g(x,y)$$

$$\frac{\partial^2 f}{\partial y\partial x}(0,0)=\lim_{y\to0}\frac{\frac{\partial F}{\partial x}(0,y)-\overbrace{\frac{\partial F}{\partial x}(0,0)}^{=0}}{y}=\lim_{y\to0}\lim_{x\to0}g(x,y)$$

$$\frac{\partial^2 f}{\partial x\partial y}(0,0)=\lim_{x\to0}\lim_{y\to0}g(x,y)$$ $\Rightarrow$ כלומר, סדר השאפת הגבולות משנה.

למשל - $g(x,y)=\begin{cases}\frac{x^2-y^2}{x^2+y^2} & ,x^2+y^2\neq0\\ 0 & ,x^2+y^2=0\end{cases}$ כאן שונה של הגבולים.

$$\lim_{y\to0}\lim_{x\to0}g(x,y)=-1$$

$$\Rightarrow\quad\frac{\partial^2 f}{\partial x\partial y}=+1\ ,\quad\frac{\partial^2 f}{\partial y\partial x}=-1$$

$$\lim_{x\to0}\lim_{y\to0}g(x,y)=+1$$

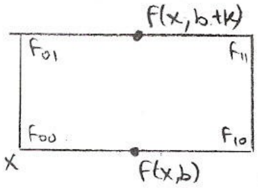
בגלל ששינינו את סדר הגבולות,

נניח ש- $\frac{\partial f}{\partial x_j}$ , $\frac{\partial f}{\partial x_i}$ , $\frac{\partial^2 f}{\partial x_i \partial x_j}$ קיימת בסביבה $p \in U$ .

$U_p$ של $p$ , ו- $\frac{\partial^2 f}{\partial x_i \partial x_j}$ רציפה ב- $U_p$ , אז קיימת $\frac{\partial^2 f}{\partial x_i \partial x_j}(p) = \frac{\partial^2 F}{\partial x_j x_i}(p)$ .

$n=2$ , $x_j = x$ , $x_i = y$ , $p=(a,b)$ .

$\frac{\partial^2 f}{\partial y \partial x}$ קיימת ורציפה בסביבה של $(a,b)$ $\overset{?}{\Longleftarrow}$ $\exists \frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$




נגדיר $g(x) = f(x,b+k) - f(x,b)$

$$f_{11} - f_{01} - f_{10} + f_{00} = g(a+h) - g(a) \underset{\text{Lagrange}}{=} h g'(\xi) \; = $$

$$= h\left(\frac{\partial f}{\partial x}(\xi, b+k) - \frac{\partial f}{\partial x}(\xi, b)\right) \underset{\text{Lagrange}}{=} hk \frac{\partial^2 f}{\partial y \partial x}(\xi, \eta) \; =$$

$a < \xi < a+h$ $\qquad$ $b < \eta < b+k$

$$\underset{\text{רציפות של } \frac{\partial^2 f}{\partial y \partial x}}{=} hk\left(\frac{\partial^2 f}{\partial y \partial x}(a,b) + o(1)\right)$$

אם $\forall \varepsilon \; \exists \delta : h^2 + k^2 < \delta^2$ אז -

$$\left| \frac{f_{11} - f_{01} - f_{10} + f_{00}}{hk} - \frac{\partial^2 f}{\partial y \partial x}(a,b) \right| < \varepsilon$$

$\underline{k \to 0}$ :

$$\frac{f_{11} - f_{10}}{k} = \frac{f(a+h, b+k) - f(a+h, b)}{k} \xrightarrow[k \to 0]{} \frac{\partial f}{\partial y}(a+h, b)$$

כלומר -

$$\frac{f_{11} - f_{10}}{k} \xrightarrow[k \to 0]{} \frac{\partial f}{\partial y}(a+h, b)$$

$$\frac{f_{01} - f_{00}}{k} \xrightarrow[k \to 0]{} \frac{\partial f}{\partial y}(a,b)$$

$$\left| \frac{1}{h}\left[\frac{\partial f}{\partial y}(a+h, b) - \frac{\partial f}{\partial y}(a,b)\right] - \frac{\partial^2 f}{\partial y \partial x}(a,b) \right| < \varepsilon$$

מתקיים $|h| < \delta$ ש $\exists \delta$ $\forall \varepsilon > 0$

$\Longleftrightarrow$

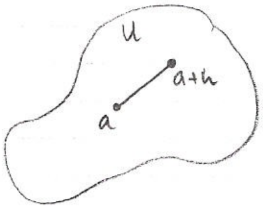
$$\underbrace{\exists \lim_{h \to 0} \frac{1}{h}\left(\frac{\partial f}{\partial y}(a+h, b) - \frac{\partial f}{\partial y}(a,b)\right)}_{\exists \frac{\partial^2 f}{\partial x \partial y}(a,b)} = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

הגדרה: $f\in C^k(U)$ אם כל נגזרות חלקיות $\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}$ קיימות ורציפות ב-U.

אז $C^\infty(U)=\bigcap_{k\geq 1} C^k(U)$.

משפט: אם $f\in C^k(U)$ אז $\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}$ לא תלויה בהחלפת סדר המשתנים.

הוכחה: ודוק (משפט 1 + אינדוקציה על n)




$f:U\to\mathbb{R}^1$ , $[a,a+h]\subset U$ , $f\in C^{m+1}(U)$

נגדיר: $0\leq t\leq 1$ , $\varphi(t)=f(a+th)$

$\varphi\in C^{m+1}[0,1]$

$$\varphi(0)=f(a)$$

$$\varphi(1)=f(a+h)$$

$$\varphi(1)=\sum_{k=0}^{m}\frac{\varphi^{(k)}(0)}{k!}\cdot 1^k+\underbrace{R_{m+1}(\varphi)}_{\text{שארית}}$$

$$f(a+h)=\;??\;+\text{שארית}\qquad h=\begin{pmatrix}h^1\\ \vdots\\ h^n\end{pmatrix}$$

$$\varphi'(t)=\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}(a+th)h^j$$

$$\varphi''(t)=\sum_{j=1}^{n}\underbrace{\left(\frac{\partial}{\partial t}\frac{\partial f}{\partial x_j}(a+th)\right)}_{=\sum_{i=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}(a+th)h^i}h^j=\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}(a+th)h^ih^j$$

$\vdots$

$$\varphi^{(k)}(t)=\sum_{i_1,\dots,i_k=1}^{n}\frac{\partial^k f}{\partial x_{i_1}\dots\partial x_{i_k}}(a+th)h^{i_1}\dots h^{i_k}$$

בדיקה באינדוקציה על k.

$\underline{t=0}\Rightarrow$

$$\varphi^{(k)}(0)=\sum_{i_1,\dots,i_k=1}^{n}\frac{\partial^k f}{\partial x_{i_1},\dots,\partial x_{j_1}}(a)h^{i_1}\dots h^{i_k}$$

$$f(a+h)=\underbrace{\sum_{k=0}^{m}\frac{1}{k!}\sum_{i_1,\dots,i_k=1}^{n}\frac{\partial^k f}{\partial x_{i_1}\dots\partial x_{i_k}}(a)h^{i_1}\dots h^{i_k}}_{=P_m(h;a,f)}+\text{שארית}\;\textcircled{=}$$

פולינום טיילור של f סדר m בנקודה a.

$$\textcircled{=}\;\underbrace{f(a)}_{k=0}+\underbrace{\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(a)h^i}_{k=1}+\underbrace{\frac{1}{2}\sum_{i_1,i_2=1}^{n}\frac{\partial^2 f}{\partial x_{i_1}\partial x_{i_2}}(a)h^{i_1}h^{i_2}}_{k=2}+\dots$$

$$f(a+h)=P_m(h;a,f)+R_{m+1}$$

$$|R_{m+1}| \le \frac{N_{m+1}}{(m+1)!}\left(\sqrt{n}\,|h|^{m+1}\right)$$ : Lagrange לפי משפט

$$N_{m+1} = \sup_{x\in U} \max_{i_1,\dots,i_{m+1}} \left|\frac{\partial^{m+1} f}{\partial x_{i_1}\dots\partial x_{i_{m+1}}}(x)\right|$$ כאשר -

$$|R_{m+1}| = \frac{1}{(m+1)!}\left|\varphi^{(m+1)}(\xi)\right| = \frac{1}{(m+1)!}\cdot\sum_{i_1,\dots,i_{m+1}=1}^{n} \frac{\partial^{m+1} f}{\partial x_{i_1}\cdots\partial x_{i_{m+1}}}(a+\xi h)h^{i_1}\dots h^{i_{m+1}} \le$$

לפי $\varphi^{(k)}(t)$
$t=\xi$
$k=m+1$

$$\le \frac{1}{(m+1)!}\sum_{i_1,\dots,i_k=1}^{n} N_{m+1}|h^{i_1}|\dots|h^{i_{m+1}}| = \frac{N_{m+1}}{(m+1)!}\left(\sum_{j=1}^{n}|h^j|\right)^{m+1}$$

$$\left( \left(\sum_{i=1}^{n} a_i\right)^2 = \left(\sum_{i_1=1}^{n} a_{i_1}\right)\left(\sum_{i_2=1}^{n} a_{i_2}\right) = \sum_{i_1,i_2=1}^{n} a_{i_1}a_{i_2} \right)$$

$$\sum_{i=1}^{n}|h^i| \le \sqrt{h\cdot\sum_{i=1}^{n}(h^i)^2} = \sqrt{n}\,|h|$$

(אי-שוויון Cauchy-Schwartz: $\sum \alpha_i\beta_i \le \sqrt{\sum\alpha_i^2\sum\beta_i^2}$, $\alpha_i,\beta_i>0$, $\beta_i=|h^i|,\ \alpha_i=1$)

$$\Longrightarrow \quad |R_{m+1}| \le \frac{N_{m+1}}{(m+1)!}\left(\sqrt{n}\,|h|\right)^{m+1}$$

$F\in C^m(U)$, אז - $F(a+h) = P_m(h;a,F) + o(|h|^m)$, $h\to 0$.

צריך ה- m-1 פעמים, Lagrange.

$$f(a+h) = P_{m-1}(h) + R_m = P_{m-1}(h) + \frac{1}{m!}\sum_{i_1,\dots,i_k=1}^{n}\frac{\partial^m F}{\partial x_{i_1}\cdots\partial x_{i_m}}(a+\xi h)\cdot h^{i_1}\dots h^{i_m} =$$
$0<\xi<1$

רציפות של $\partial^m F$

$$= P_{m-1}(h) + \frac{1}{m!}\sum_{i_1,\dots,i_m=1}^{n}\left(\frac{\partial^m F}{\partial x_{i_1}\dots\partial x_{i_m}}(a) + o(1)\right)h^{i_1}\dots h^{i_m} =$$

$$= \underbrace{P_{m-1}(h) + \frac{1}{m!}\sum_{i_1,\dots,i_m=1}^{n}\frac{\partial^m F}{\partial x_{i_1}\dots\partial x_{i_m}}(a)\,h^{i_1}\dots h^{i_m}}_{=P_m(h)} + \text{שארית}$$

$$|\text{שארית}| \le \frac{1}{m!}\,o(1)\cdot\underbrace{\sum_{i_1,\dots,i_m=1}^{n}|h^{i_1}|\cdot\ldots\cdot|h^{i_m}|}_{\le(\sqrt{n}\,|h|)^m} \le o(1)|h|^m = o(|h|^m)$$

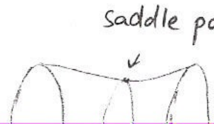
$\nabla F(a)=0$, $F\in C^2(U)$

$$F(a+h) = F(a) + \underbrace{\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(a)h^i}_{=0} + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 F}{\partial x_i\partial x_j}(a)h^ih^j + o(|h|^2) =$$

$$= F(a) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 F}{\partial x_i\partial x_j}(a)h^ih^j + o(|h|^2)$$

<u>Local max.</u> - $F(a+h)\le F(a)$ לכל $h$, <u>local min.</u> - $F(a+h)\ge F(a)$

saddle point



Saddle point